Optimal Liquidation Strategies under a Transient Price Impact Model

A Discrete-time Analysis with select Continuous-time Representation

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A Brief Review of Optimal Liquidation Problems

Given an inventory of shares, how should an agent trade in order to maximize her risk-adjusted utility?

Unmitigated policies contend with impact and uncertainty risk:

- **Impact Risk**: Trading too quickly or in large block sizes carries a penalty as per the laws of supply/demand
- **Uncertainty Risk**: Trading too slowly compromises forecast accuracy and faces pressure from risk aversion.

Seminal work from **Almgren & Chriss (2001)** introduced temporary and permanent price impacts induced by the rate of trade, extended to volume impacts by **Obizhaeva & Jiang Wang (2013)** and **Bank & Voß (2018)**.

The Aim of the Work

The coupled bid/ask price model given in **Bank & Voß (2018)** becomes the subject for refinement, under which the goals are to:

1. Introduce an impact model for the bid- and ask- price capturing . . .
   - . . . directional price sensitivity to trading volumes
   - . . . transient price recovery towards an unaffected price
2. Develop a discretization framework that admits an optimal trading policy
   - with an appreciation for the economic drivers to the model
3. Extend the discrete model for continuous time representation

Price Impact Model

Consider the trading policy γ_t defined by the non-decreasing cádlág functions for the cumulative volume of purchases (+) and sales (-)

$$\gamma_t = \gamma_{0^-} + \gamma_t^+ - \gamma_t^-$$

Trading activity impacts the bid- and ask- price according to

$$dA_t = dP_t + \lambda d\gamma_t^+ - \alpha(A_t - P_t)dt$$
$$dB_t = dP_t - \lambda d\gamma_t^- - \alpha(B_t - P_t)dt$$

where

- $\lambda > 0$ is the trading impact coefficient $:\rightarrow$ **"Depth"**
- $\alpha > 0$ is the market resilience rate $:\rightarrow$ **"Resilience"**
- $(P)_{t\geq 0}$ is the fundamental price process, taken in the simplest form $dP_t = \mu dt + \sigma dW_t$

Induced Spread Dynamics

The bid/ask spread $S = A - B$ and the mid-quote $Q = (A + B)/2$ evolve according to

$$dQ_t = dP_t + \frac{1}{2}\lambda d\gamma_t - \alpha(Q_t - P_t)dt$$
$$dS_t = \lambda|d\gamma|_t - \alpha S_t dt$$

for $|d\gamma|_t = d\gamma_t^+ + d\gamma_t^-$.

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for $|d\gamma|_t = d\gamma_t^+ + d\gamma_t^-$. Define $R = 2(Q-P)$,

$$R_t = R_{0-}e^{-\alpha t} + \lambda \int_{[0,t]} e^{-\alpha(t-u)} d\gamma_u :\to \textbf{"Bias"}$$
$$S_t = S_{0-}e^{-\alpha t} + \lambda \int_{[0,t]} e^{-\alpha(t-u)} |d\gamma|_u :\to \textbf{"Tightness"}$$

Remark: Under **Bank & Voß (2018)**, $dQ_t = dP_t + \frac{1}{2}\lambda d\gamma_t$.

Liquidation Wealth Process

Under the linear transient price impact model, consider the wealth received after selling $\Delta\gamma_t^- = \gamma_t^- - \gamma_{t-}^-$ shares of the asset at $t \in [0, \tau]$

$$w_t = \int_0^{\Delta\gamma_t^-} (B_{t^-} - \lambda x) dx = B_{t^-} \Delta\gamma_t^- - \frac{\lambda}{2} |\Delta\gamma_t^-|^2$$

with purchasing treated analogously, this gives

$$dw_t = \left(B_{t^-} - \lambda \Delta\gamma_t^-\right) d\gamma_t^- - \left(A_{t^-} + \lambda \Delta\gamma_t^+\right) d\gamma_t^+$$

Remark: The wealth dynamics suggest sub-optimality for any simultaneous purchase and sale.

Statement of the Optimal Control Problem

For some risk aversion level $\beta \geq 0$, the objective is to solve for the control γ which maximizes the agent's mean-variance trade-off

$$\mathbb{E}[W_\tau] - \frac{\beta}{2}\mathbb{V}[W_\tau] \to max!$$

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When $(P_t)_{t\geq 0}$ is an ABM with constant coefficients, the objective is equivalent to finding the minimizer of $J(\cdot)$

$$J(\gamma) = \frac{\beta\sigma^2}{2}\int_0^\tau \gamma_t^2 dt + \mu \int_{[0,\tau]} t d\gamma_t + \frac{\lambda}{2}\Bigg[\frac{R_{0-}}{\lambda}\int_{[0,\tau]} e^{-\alpha t} d\gamma_t + \frac{S_{0-}}{\lambda}\int_{[0,\tau]} e^{-\alpha t}|d\gamma|_t + \int_{[0,\tau]}\int_{[0,\tau]} e^{-\alpha|t-s|} d\gamma_s^- d\gamma_t^- + \int_{[0,\tau]}\int_{[0,\tau]} e^{-\alpha|t-s|} d\gamma_s^+ d\gamma_t^+\Bigg] \to min!$$

Discrete Time Modeling

Assumption

For an agent looking to maximize her risk-adjusted liquidation wealth, trading is only allowed over an evenly spaced time grid $\Xi = \{0, h, 2h, \ldots, \tau\}$ for $h = \tau/n$.

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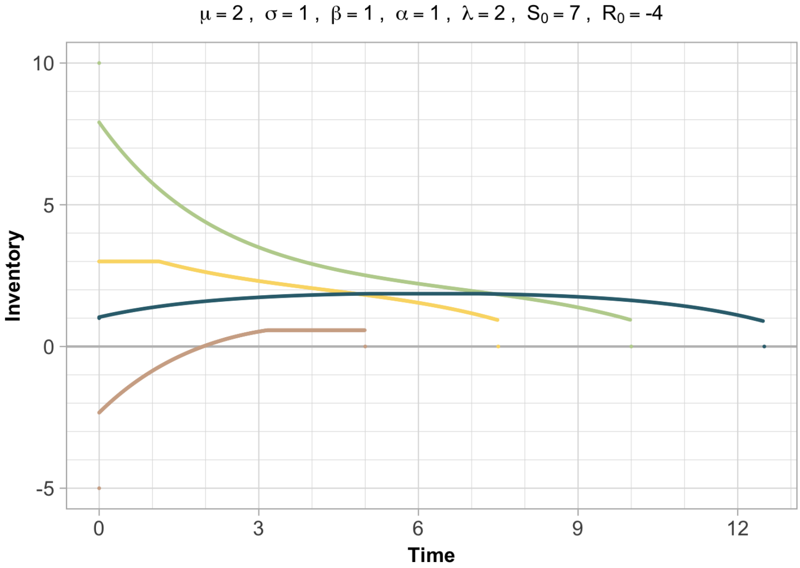
Under this framework, the minimization is now of $J_n(\gamma^n)$

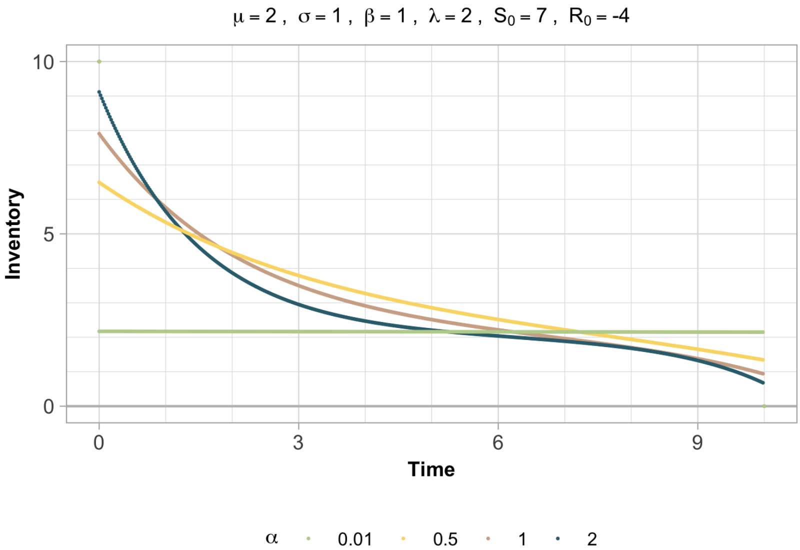
$$J_n(\gamma^n) = \frac{\beta\sigma^2 h}{2} \sum_{i=0}^{n} \Big[\sum_{j=0}^{i} (\Delta\gamma_{jh}^{+} - \Delta\gamma_{jh}^{-}) \Big]^2 + \mu h \sum_{i=0}^{n} i \left(\Delta\gamma_{jh}^{+} - \Delta\gamma_{jh}^{-} \right) + C_n(\gamma) \to min!$$

when defining the cost functional $C_n(\cdot)$

A Comparison of Policies in the Same Market



Market **Resilience** & Depth as a Policy Driver



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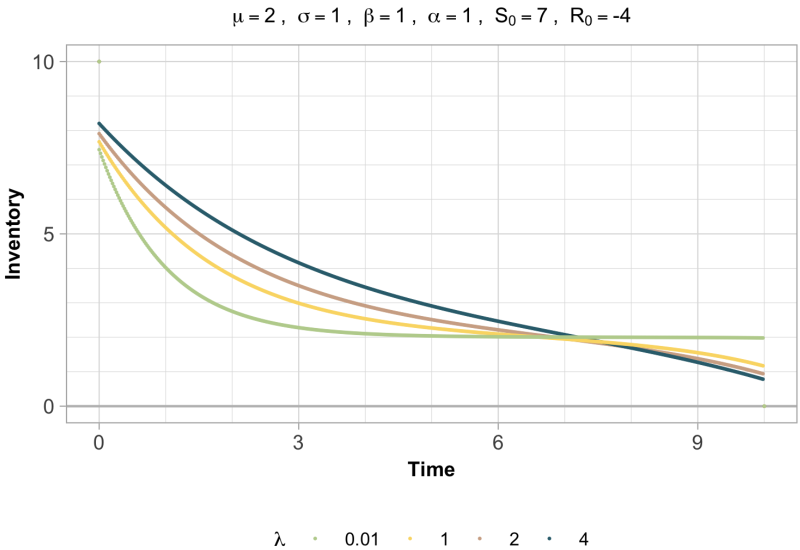
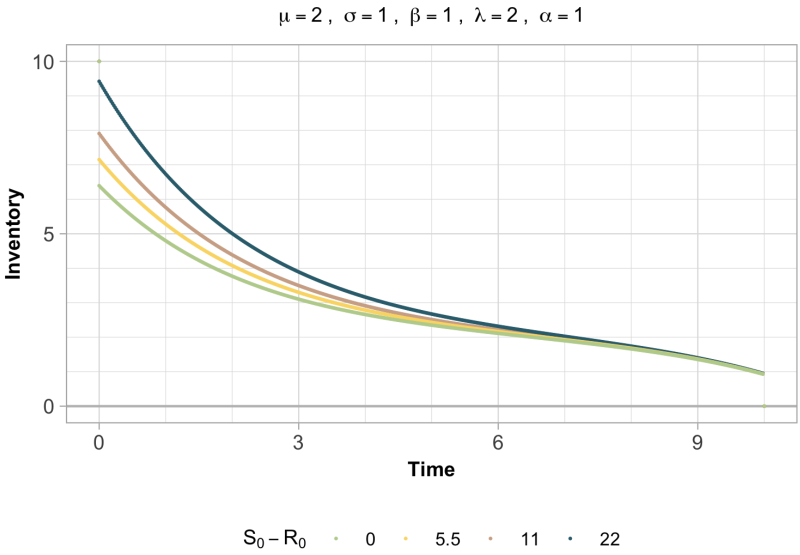


Market Bias & Tightness as a Policy Driver



A Convexity Argument

Owing to the convexity of the problem, for any two trading policies ξ and γ having $\xi_{0-} = \gamma_{0-}$, the following holds given $\epsilon \in (0,1]$

$$
\begin{aligned}
J_\tau(\xi) - J_\tau(\gamma) &\geq \lim_{\epsilon \to 0} \frac{J_\tau(\epsilon\xi + (1-\epsilon)\gamma) - J_\tau(\gamma)}{\epsilon} \\
&\geq \int_{[0,\tau]} \nabla_t^+ J_\tau(\gamma)\, (d\xi_u^+ - d\gamma_u^+) \\
&\quad + \int_{[0,\tau]} \nabla_t^- J_\tau(\gamma)\, (d\xi_u^- - d\gamma_u^-)
\end{aligned}
$$

First Order Optimality Condition

An absolutely continuous trading policy $\hat{\gamma}$ is optimal when $\nabla_t^{\pm} J_\tau(\hat{\gamma}) = 0$.

A Variational Approach for Continuous Modeling

In the spirit of **Bank & Voß (2018)**, calculate the infinite dimensional buying- and selling- subgradients for $J_\tau(\gamma)$

$$\nabla_t^{\pm} J_\tau(\gamma) = \nabla_t^{\pm} D_\tau(\gamma) + \nabla_t^{\pm} L_\tau(\gamma) + \nabla_t^{\pm} Q_\tau(\gamma)$$

where

$$\nabla_t^{\pm} D_\tau(\gamma) = \begin{cases} \pm\beta\sigma^2 \int_t^\tau (\gamma_u - \frac{\mu}{\beta\sigma^2}) du & \text{if } \beta > 0 \\ \mu & \text{otherwise} \end{cases}$$

$$\nabla_t^{\pm} L_\tau(\gamma) = \frac{1}{2}(S_{0-} \pm R_{0-}) e^{-\alpha t}$$

$$\nabla_t^{\pm} Q_\tau(\gamma) = \frac{1}{2} e^{-\alpha(\tau - t)} \Big[(S_\tau \pm R_\tau) - (S_{0-} \pm R_{0-}) e^{-\alpha\tau} \Big] + \alpha \int_t^\tau \Big[(S_u \pm R_u) - (S_{0-} \pm R_{0-}) e^{-\alpha u} \Big] e^{-\alpha(u-t)} du$$

Dynamics of the Continuously Trading Policy

Assume that γ is absolutely continuous over $(a, b) \subset (0, \tau)$. Following a routine of setting $\nabla_t^{\pm} J_\tau(\gamma) = 0$ and differentiating, as per **Bank & Voß (2018)**, it can be shown

$$\ddot{\gamma}_t^- = \frac{\alpha^2 \beta \sigma^2}{2\alpha\lambda + \beta\sigma^2}\left(\frac{\mu}{\beta\sigma^2} - \gamma_t\right)$$

With the general form

$$\gamma_t^- = c_+ e^{\theta t} + c_- e^{-\theta t} + \gamma_0 - \frac{\mu}{\beta\sigma^2}$$

Having a signed difference for γ_t^+.

Recap and Relevance

- Refinement on the bid/ask model supporting a more robust economic intuition
- Simplistic modeling of a stochastic control problem as a QP
- Alternative approach to the Hamilton-Jacobi-Bellman PDE

$$
\begin{aligned}
0 = max\Bigg\{ (\mu c + 1)\frac{\partial J}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 J}{\partial p^2} - \alpha r\frac{\partial J}{\partial r} - \alpha\frac{\partial J}{\partial s}, \\
\lambda\frac{\partial J}{\partial r} - \frac{1}{2}r + \frac{\partial J}{\partial c} + \lambda\frac{\partial J}{\partial s} - \frac{1}{2}s, \\
-\lambda\frac{\partial J}{\partial r} + \frac{1}{2}r - \frac{\partial J}{\partial c} + \lambda\frac{\partial J}{\partial s} - \frac{1}{2}s, \\
\max_{\Delta c}\left\{ \Delta J - \frac{r}{2}\Delta c - \frac{2}{2}|\Delta c| - \frac{\lambda}{2}|\Delta c|^2 \right\} \Bigg\}
\end{aligned}
$$

Model Extensions

- Alternative models for the LOB
  - Time-varying market resilience $\alpha(t)$
  - Non-linear price impacts
- Increase dimensionality
  - $(A)_{t\geq 0}$, $(B)_{t\geq 0}$ and $(P)_{t\geq 0}$ become p-dimensional
  - $\Lambda \in \mathbb{R}^{p\times p}$ and $\alpha \in \mathbb{R}^{p\times p}$
- Allow for adverse impacts from competing agents ξ
  - Setup and solve the Nash-equilibrium

$$dA_t = dP_t + \Lambda(d\gamma_t^+ + d\xi_t^+) - \alpha(A_t - P_t)dt$$
$$dB_t = dP_t - \Lambda(d\gamma_t^- + d\xi_t^-) - \alpha(B_t - P_t)dt$$

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